GROUPS GENERATED BY POWERS OF DEHN TWISTS

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Abstract. We classify groups generated by powers of 2 Dehn twists which are 1) free or 2) have no "unexpected" reducible elements. We give some sufficient conditions in the case of groups generated by powers of $h \geq 3$ twists.

Key words: Mapping class group, Dehn twist, pseudo-Anosov.

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§0. Introduction

Let S be a punctured or non-punctured oriented surface. For (the isotopy class of)[3] a simple closed curve c on S let D_c denote the right-handed Dehn twist about c. Let (c_1, c_2) denote the geometric intersection number of simple closed curves c_1, c_2. Let $M(S)$ be the mapping class group of S. Let $\mathbb{F}_h$ be the free group on h generators. For a set of h simple closed curves $A = \{a_1, \cdots a_h\}$ and positive integers $n_1, \cdots n_h$ we study the group $G = \langle D_{a_1}^{n_1}, \cdots D_{a_h}^{n_h} \rangle$, and ask the question whether $G \cong \mathbb{F}_h$. It is well known that $G \cong \mathbb{F}_h$ if $(a_i, a_j) \neq 0$ for $i \neq j$ and n_i are large, for all i, j (See for example [I]).

In the case $h = 2$ we will give a complete answer, i.e., $G \ncong \mathbb{F}_2$ if and only if

$$((a_1, a_2), \{n_1, n_2\}) = (0, \{*, *\}), (1, \{1, 1\}), (1, \{1, 2\}), \text{ or } (1, \{1, 3\})$$

(See Theorem 2.4).

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[3]We will usually drop this in the rest of this paper for brevity.

It should be noticed that in the case of two curves a, b filling up a closed surface this was done by Thurston as a method to construct pseudo-Anosov elements; i.e., he showed that $\langle D_a, D_b\rangle$ is free and consists of pseudo-Anosov elements besides powers of conjugates of the generators [FLP]. Our methods are completely different and elementary, and are only based on how the geometric intersection pairing behaves under Dehn twists.

In the case when $h \geq 3$, we give some sufficient conditions for $G \cong \mathbb{F}_h$. To motivate our condition, look at the case $\Gamma = \langle D_{a_1}, D_{a_2}, D_{a_3}\rangle$, and assume $a_3 = D_{a_1}(a_2)$. Now $D_{a_3} = D_{a_1} D_{a_2} D_{a_1}^{-1}$, so $G \ncong \mathbb{F}_3$. But notice that $(a_1, a_3) = (a_1, a_2)$ and $(a_2, a_3) = (a_1, a_2)^2$. This shows that the set $I = \{(a_i, a_j) \mid i \neq j\}$ is "spread around". It turns out this is in a sense the only obstruction for $\Gamma \cong \mathbb{F}_h$:

Theorem 0.1. *Suppose* $\Gamma = \langle D_{a_1}, \cdots D_{a_h}\rangle$, *and let* $m = \min I$ *and* $M = \max I$, *where* $I = \{(a_i, a_j) \mid i \neq j\}$. *Then* $\Gamma \cong \mathbb{F}_h$ *if* $M \leq m^2/6$.

We will prove a more general version of this (see Theorem 3.2).

The second question we ask about the group G is that to what extent the elements of G are pseudo-Anosov? a mapping class f is called pseudo-Anosov if $f^n(c) \neq c$ for all non-trivial simple closed curves and $n > 0$. Let $A = \{a_1, a_2, \cdots, a_h\}$ be a set of simple closed curves on S. The surface filled by A, denoted S_A is a regular neighborhood N of $\cup A$ together with the components of $S - N$ which are discs with 0 or 1 punctures. We say that A fills up S if $S_A = S$. Let $f = D_{c_1}^* \cdots D_{c_k}^*$ be a cyclically-reduced word in the generators of G. Let $supp(f) = S_{\{c_1, \cdots, c_k\}}$. Then we say f is relatively pseudo-Anosov if f is either the identity or pseudo-Anosov in $M(supp(f))$. If $g = hfh^{-1}$ with f cyclically-reduced, define $supp(g) = h(supp(f))$. The group G is relatively pseudo-Anosov if every word in generators of G is relatively pseudo-Anosov. Notice that a Dehn twist D_{a_i} is relatively pseudo-Anosov since its support is a twice-punctured sphere. Intuitively, a group G is relatively pseudo-Anosov if no element of G has "unexpected reducibility". It is well known that for n_i large, G is relatively pseudo-Anosov for any choice of curves a_i.

In the case $h = 2$ again we are able to give a complete answer. G is *not* relatively pseudo-Anosov if and only if

$$((a_1, a_2), \{n_1, n_2\}) = (1, \{1, 1\}), (1, \{1, 2\}), (1, \{1, 3\}), \text{ or } (2, \{1, 1\})$$

(see Theorem 2.10).

In the case of $h \geq 3$ we get some bounds on the powers of the generators sufficient for the group G to be relatively pseudo-Anosov in the case each pair of the curves fill up the surface (see Theorem 3.6). We think that this should be answered in a more satisfactory way. This could be the subject of a forthcoming paper.

Question 0.2. *Under what conditions* $\Gamma = \langle D_{a_1}, \cdots, D_{a_h} \rangle$ *is relatively pseudo-Anosov?*

In §1 we go over basic fact about Dehn twists and geometric intersection pairing and different kinds of ping-pong arguments we are going to use.

In §2 we determine exactly when a group generated by powers of 2 Dehn twists is free or relatively pseudo-Anosov.

In §3 we generalize the arguments for the case of $h \geq 3$ powers of Dehn twists.

§4 is devoted to the simplest case of a group generated by powers of 3 Dehn twists, i.e., when the curves are the $(1, 0)$, $(0, 1)$, and the $(1, 1)$ curve on the torus. This group was studied previously in a more general setting (see [BM], [Sch]). We show how to apply our methods in this case, and moreover we determine which ones do not contain surprising parabolic elements.

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§1. Basics

For two simple closed curves a, b let (a, b) denote their geometric intersection number. For a set of simple closed curves $A = \{\alpha_1, \cdots, \alpha_n\}$ and a simple closed curve (or measured lamination) x put

$$||x||_A = \sum_{i=1}^{n} (x, \alpha_i).$$

Let D_a be the (right-handed) Dehn twist in curve a. The following lemma is proved in [FLP].

Lemma 1.1. *For simple closed curves* a, b, c, *and* $n \geq 0$,

$$|(D_a^{\pm n}(b), c) - n(a, b)(a, c)| \leq (b, c). \spadesuit$$

Lemma 1.2. *(The ping-pong lemma; PPL) Let G be a group generated by $f_1, \cdots, f_n$, $n \geq 2$. Suppose G acts on a set Z. Assume that there are n non-empty mutually disjoint subsets $X_1, \cdots, X_n$ of Z such that $f_i^{\pm n}(\cup_{j \neq i} X_j) \subset X_i$, for all $1 \leq i \leq n$. Then $G \cong \mathbb{F}_n$.*

Proof. First notice that reduced words of the form $w = f_1^* f_i^* \cdots f_j^* f_1^*$ (*'s are non-zero integers) are not the identity because $w(X_2) \cap X_2 = \emptyset$. But any non-trivial word in $f_1^*, \cdots, f_n^*$ is conjugate to a w of the above form. ♠.

Lemma 1.3. *(ping-pong in the world trade center; PPWTC) Let G be a group generated by $f_1, \cdots, f_n$. Suppose G acts on a set Z, and there is a function defined $||.|| : Z \to \mathbb{R}_{\geq 0}$, with the following properties: There are n non-empty mutually disjoint subsets $X_1, \cdots, X_n$ of Z such that $f_i^{\pm n}(Z - X_i) \subset X_i$ and for any $x \in Z - X_i$, we have $||f^{\pm n}(x)|| > ||x||$ for all $n > 0$. Then $G \cong \mathbb{F}_n$. Moreover, the action of every $g \in G$ which is not conjugate to some power of some f_i on Z has no periodic points.*

Proof. Any reduced word in $f_1^*, \cdots, f_n^*$ (*'s denote non-zero integers) is conjugate to a reduced word $w = f_1^* \cdots f_1^*$. To show that $w \neq id$ notice that if $x_1 \in Z - X_1$, then $w(x_1) \in X_1$, therefore $w(x_1) \neq x_1$. To prove the last assertion, notice that it's enough to show the claim with "fixed points" replaced by "periodic points". Any element of G which is not conjugate to a power of some f_i is conjugate to some word of the form $w = f_j^* \cdots f_i^*$ with $i \neq j$. Now suppose $w(x) = x$. First assume $x \in Z - X_i$. Then by assumption $||w(x)|| > ||x||$ which is impossible. If on the other hand, $x \in X_i$ and $w(x) = x$, then $w^{-1}(x) = f_i^* \cdots f_j^*(x) = x$. But again by assumption $||w^{-1}(x)|| > ||x||$, which is a contradiction. ♠

Sometimes it is hard or impossible to come up with subsets $X_1, \cdots, X_n$ Satisfying the assumptions of PPWTC. Here is a weaker version we are going to use.

Lemma 1.4. *(Weak ping-pong; WPP) Let G be a group generated by $f_1, \cdots, f_n$. Suppose G acts on a set $Z_1 \supset Z$, and there is a function defined $||.|| : Z \to \mathbb{R}_{\geq 0}$, with the following properties: There are n non-empty mutually disjoint subsets $X_1, \cdots, X_n$ of Z such that $f_i^{\pm n}(Z - X_i) \subset X_i$ and for any $x \in Z - X_i$, we have $||f^{\pm n}(x)|| > ||x||$ for all $n > 0$. Also, assume that there exists an n_0 such that for every irreducible word $w = f_{k_1}^* \cdots f_{k_{n_0}}^*$, we have $w(Z_1 - Z) \subset X_{k_1}$. Then $G \cong \mathbb{F}_n$. Moreover, the action of every $g \in G$ which is not conjugate to some power of some f_i on Z has no periodic points. Also, if there is some*

$f \in G$ and $x \in Z_1 - Z$ such that $f^n(x) = x$, then $f = w^k$ for some word w of length $\leq n_0$ and $k \geq 1$.

Proof. Easily obtained by altering the proof of 1.3. ♠

Remark 1.5. In application of WPP, one must notice that, under the assumptions of the lemma, one may replace the assumption $w(Z_1 - Z) \subset X_{k_1}$ with $w_j(Z_1 - Z) \subset X_{k_j}$ for some subword of w of the form $w_j = f^*_{k_j} \cdots f^*_{k_{n_0}}$.

Lemma 1.6. *(Cauchy-Schwartz inequality) If simple closed curves $A = \{\alpha_1, \cdots, \alpha_n\}$ fill up S, then for any two simple closed curves c_1, c_2,*

(1) *If S is a closed surface, then*

$$(c_1, c_2) \leq ||c_1||_A ||c_2||_A,$$

(2) *If S has punctures*

$$(c_1, c_2) \leq 2||c_1||_A ||c_2||_A.$$

Proof. $S - \cup A$ is a union of discs or once punctured discs. One can put c_1 and c_2 in a transversal position such that each segment of $c_1 - \cup A$ and $c_2 - \cup A$ intersect at most once in non-punctured discs, and intersect at most twice in the punctured disks, while no three curves in $A \cup \{c_1, c_2\}$ pass through the same point. Therefore for each pair of segments in $c_1 - (\cup A)$ and $c_2 - (\cup A)$ one gets at most 1 (resp. 2) intersection point(s) for closed (resp. punctured) S, hence the inequality. ♠

§2. Groups generated by powers of two Dehn twists

Let a, b be simple closed curves on S with $(a, b) = m > 0$. Put $A = \{a, b\}$. Consider the sets of isotopy classes of simple closed curves

$$X = \{x \mid ||x||_A > 0 \}.$$

Then for $\lambda \in (0, \infty)$ set

$$N_{a,\lambda} = \{x \in X \mid \ (x, a) < \lambda(x, b)\},$$

$$N_{b,\lambda^{-1}} = \{x \in X \mid \ \lambda(x, b) < (x, a)\}.$$

Notice that $a \in N_{a,\lambda}$ and $b \in N_{b,\lambda^{-1}}$, and $N_{a,\lambda} \cap N_{b,\lambda^{-1}} = \emptyset$.

Lemma 2.1.

1. $D_a^{\pm n}(N_{b,\lambda^{-1}}) \subset N_{a,\lambda}$ *if* $mn \geq 2\lambda^{-1}$.
2. *If* $mn \geq 2\lambda^{-1}$ *and* $x \in N_{b,\lambda^{-1}}$, *then* $||D_a^{\pm n}(x)||_A > ||x||_A$.
3. $D_b^{\pm n}(N_{a,\lambda}) \subset N_{b,\lambda^{-1}}$ *if* $mn \geq 2\lambda$.
4. *If* $mn \geq 2\lambda$ *and* $x \in N_{a,\lambda}$, *then* $||D_b^{\pm n}(x)||_A > ||x||_A$.

Proof. Suppose $x \in N_b$ and $n > 0$. Then

$$\begin{aligned}(D_a^{\pm n}(x), b) &\geq mn(x,a) - (x,b)\\ &> (mn - \lambda^{-1})(x,a)\\ &= (mn - \lambda^{-1})(D_a^{\pm n}(x), a).\end{aligned}$$

This shows that $D_a^{\pm n}(N_b) \subset N_a$ if $mn \geq 2\lambda^{-1}$, which proves (1). By symmetry we immediately get (3). Now notice that for $x \in N_b$,

$$\begin{aligned}||D_a^{\pm n}(x)||_A &= (D_a^{\pm n}(x), a) + (D_a^{\pm n}(x), b)\\ &\geq (x,a) + mn(x,a) - (x,b)\\ &> (1 + mn - \lambda^{-1})(x,a)\\ &= (1 + mn - \lambda^{-1})(1+\lambda)^{-1}((x,a) + \lambda(x,a))\\ &> (1 + mn - \lambda^{-1})(1+\lambda)^{-1}(\lambda(x,b) + \lambda(x,a))\\ &= \lambda(1 + mn - \lambda^{-1})(1+\lambda)^{-1}||x||_A.\end{aligned}$$

But $\lambda(1 + mn - \lambda^{-1})(1+\lambda)^{-1} \geq 1$ iff $mn \geq 2\lambda^{-1}$, which proves (2), and by symmetry (4).
♠

Theorem 2.2. *For Two simple closed curves* a, b *on a surface* S,

1. *If* $(a,b) \geq 2$, *the group* $\langle D_a, D_b \rangle \cong \mathbb{F}_2$ *(the free group on two generators).*
2. *If* $(a,b) = 1$, *the group* $\langle D_a^{n_1}, D_b^{n_2} \rangle \cong \mathbb{F}_2$ *for* $n_1, n_2 \geq 2$.
3. *If* $(a,b) = 1$ *then* $\langle D_a, D_b^n \rangle \cong \mathbb{F}_2$ *if* $n \geq 4$.

Proof. First consider the case $\lambda = 1$. The group generated by D_a and D_b acts on X, and if $nm \geq 2$ one can apply PPL. But if $m \geq 2$ this condition is automatically satisfied, and if $m = 1$ one can apply it if both exponents are ≥ 2. This proves (1),(2).

To prove (3), consider the case $\lambda = 2$. Lemma 2.1 shows that $D_a^{\pm n}(N_{b,1/2}) \subset N_{a,2}$ if $n \geq 1$, and $D_b^{\pm n}(N_{a,2}) \subset N_{b,1/2}$ if $n \geq 4$. This finishes the proof by PPL. ♠.

Corollary 2.3. *For two simple closed curves a, b, the group $\langle D_a, D_b \rangle \cong \mathbb{F}_2$ iff $(a, b) \geq 2$.*

Proof. If $(a, b) = 0$,the twists D_a, D_b commute. If $(a, b) = 1$ then the twists D_a, D_b satisfy the braid relation $D_a D_b D_a = D_b D_a D_b$. If $(a, b) \geq 2$, use Theorem 2.2. ♠

Theorem 2.4. *Let $A = \{a, b\}$ be a set of two simple closed curves on a surface S and $n_1, n_2 > 0$ be integers. Put $G = \langle D_a^{n_1}, D_b^{n_2} \rangle$. The following conditions are equivalent:*

(1) *$G \cong \mathbb{F}_2$.*
(2) *Either $(a, b) \geq 2$, or $(a, b) = 1$ and*

$$\{n_1, n_2\} \notin \{\{1, 1\}, \{1, 2\}, \{1, 3\}\}.$$

Proof. We already saw that (2) implies (1). To prove (1) implies (2), we must show that for $(a, b) = 1$, the groups $\langle D_a, D_b^n \rangle$ are not free for $n = 1, 2, 3$. Let's denote D_a by a and D_b by b for brevity. The case $n = 1$ is non-free because $aba = bab$. Now consider the case $n = 2$. Observe that

$$(ab^2)^2 = ab^2ab^2 = ab(bab)b = ab(aba)b = (ab)^3,$$

so $(ab^2)^4 a = (ab)^6 a = a(bab)(aba)(bab)(aba) = a(aba)(bab)(aba)(bab) = a(ab)^6 = a(ab^2)^4$. Which gives the relation $(ab^2)^4 a = a(ab^2)^4$ in $\langle a, b^2 \rangle$. In the case $n = 3$, notice that $(ab^3)^3 = ab^3ab^3ab^3 = ab^2(bab)b(bab)b^2 = ab^2abababab^2 = ab(bab)(aba)(bab)b = ab(aba)(bab)(aba)b = (ab)^6$. Therefore we get the relation $(ab^3)^3 a = a(ab^3)^3$. ♠

In the rest of this section we try to make sense out of the question "which elements in $\langle D_a, D_b \rangle$ are pseudo-Anosov?"

Theorem 2.5. *Let a, b be simple closed curves. Put $A = \{a, b\}$.*

(1) *If $(a, b) \geq 3$, then the group $\langle D_a, D_b \rangle$ is pure. Moreover, each element is either conjugate to a power of D_a or D_b, or restricts to a pseudo-Anosov element of S_A, and is the identity on $S - S_A$.*

(2) *If* $(a,b) = 2$ *then the same conclusion of (1) holds for* $\langle D_a, D_b^n \rangle$ *and* $\langle D_a^n, D_b \rangle$ *for* $n \geq 2$.

(3) *If* $(a,b) = 1$ *then the same conclusion of (1) holds for* $\langle D_a^{n_1}, D_b^{n_2} \rangle$ *and* $\langle D_a^{n_2}, D_b^{n_1} \rangle$ *for* $n_1 \geq 2$ *and* $n_2 \geq 3$.

(4) *If* $(a,b) = 1$ *then the same conclusion of (1) holds for* $\langle D_a^{n_1}, D_b^{n_2} \rangle$ *and* $\langle D_a^{n_2}, D_b^{n_1} \rangle$ *for* $n_1 \geq 1$ *and* $n_2 \geq 5$.

An element $f \in M(S)$ is called *pure* (cf. [I]) if for any simple closed curve c, $f^n(c) = c$ implies $f(c) = c$. In other words, by Thurston classification, there is a finite (possibly empty) set C of disjoint simple closed curves such that f fixes all curves in C, all components of $S - \cup C$, and is either identity or pseudo-Anosov on each such component. A subgroup of $M(S)$ is called pure if all elements of it are pure. Ivanov showed that $M(S)$ is virtually pure, i.e., it has pure subgroups of finite index. Namely, $ker(M(S) \to H_1(S, m\mathbb{Z}))$ for $m \geq 3$.

Proof. Let $\lambda \in (1, 3/2)$ be an irrational number. Then $X = N_{a,\lambda} \cup N_{b,\lambda^{-1}}$. Now if $mn \geq 2\lambda^{-1}$ Parts (1) and (2) of Lemma 2.1 are satisfied, and if $mn \geq 2\lambda$ parts (3),(4), and we can use PPWTC. But if $m \geq 3$, n can be as low as 1 in both cases. If $m = 2$, n_2 has to be at least 2. If $m = 1$, we must have $n_1 \geq 2$ and $n_2 \geq 3$. PPWTC shows that no element in the corresponding groups can have a periodic curve which intersects a or b. This means that all such elements are pseudo-Anosov on S_A. On the other hand, all elements leave all the points fixed outside a regular neighborhood of $a \cup b$. To prove (4), take $\lambda \in (2, 5/2)$ an irrational number. The first condition is $n_1 \geq 1$ and the second one is $n_2 \geq 5$. ♠

Corollary 2.6. In Theorem 2.5, the groups in each case are relatively pseudo-Anosov.

2.7 Remark. Most of these pure groups are not contained in the ones discovered by Ivanov, i.e., of the form $ker(M(S) \to H_1(S, t\mathbb{Z}))$, $t \geq 3$, if at least one of the simple closed curves a, b is non-separating.

This in particular proves

Corollary 2.8. *(Thurston; see [FLP] etc.) If* a, b *fill up the closed surface* $S = S(g)$, $g \geq 2$ *then* $\langle D_a, D_b \rangle$ *is free and all elements not conjugate to the powers of* D_a *and* D_b *are pseudo-Anosov.*

Proof. If a, b fill up S we must have $(a, b) \geq 3$. ♠

2.9 Remarks.

(i) The non-free cases $\langle D_a, D_b\rangle$, $\langle D_a, D_b^2\rangle$ and $\langle D_a, D_b^3\rangle$ where $(a, b) = 1$ are not relatively pseudo-Anosov because the maps $(D_aD_b)^6$, $(D_aD_b^2)^4$ and $(D_aD_b^3)^3$ commute with D_a (see proof of 2.4), so they all fix a, and hence are reducible on S_A.

(ii) If $(a, b) = 2$ then $G = \langle D_a, D_b\rangle$ is not relatively pseudo-Anosov. To prove this we consider two cases. The first case is when S_A is a twice or once punctured torus. In this case both a, b can be embedded in a punctured torus subsurface of S, say $a = (1, 0)$ and $b = (1, 2)$. It is easily seen by geometric inspection that D_bD_a fixes the curve $(1, 1)$ and so is not pseudo-Anosov. Case 2 is when S_A is a 4-punctured sphere. Call the punctures 1,2,3,4. Let a be the curve bounding the punctures 1,2, and b be the curve bounding the punctures 2,3. Then $D_a^{-1}D_b$ fixes the curve bounding the punctures 1,3.

(iii) If $(a, b) = 1$ then $G = \langle D_a^2, D_b^2\rangle$ is not relatively pseudo-Anosov. For $D_b^2D_a^2$ is reducible.

(iv) If $(a, b) = 1$ then $G = \langle D_a, D_b^4\rangle$ is relatively pseudo-Anosov. Notice that this is the only case which the sets $N_{a,\lambda}$ and $N_{b,\lambda^{-1}}$ and the PPWTC fail to show this fact, and we utilize WPP in this case. One can assume without loss of generality that $a = (1, 0)$ and $b = (0, 1)$ on a punctured torus. The only simple closed curves which are not in $X = N_{a,1} \cup N_{b,1}$ are $(1, 1)$ and $(1, -1)$. But both of these curves are mapped in $N_{a,1}$, $N_{b,1}$ under D_a^n $(n \geq 1)$ or D_b^k, respectively $(k \geq 4)$. This proves that they can not be periodic points for any element of G, using WPP with $n_0 = 1$.

We have shown

Theorem 2.10. *Let $A = \{a, b\}$ be a set of two simple closed curves on a surface S and $n_1, n_2 > 0$ be integers. Put $G = \langle D_a^{n_1}, D_b^{n_2}\rangle$. The following conditions are equivalent:*

(1) *G is relatively pseudo-Anosov.*

(2) *Either $(a, b) \geq 3$, or $(a, b) = 2$ and $(n_1, n_2) \neq (1, 1)$, or $(a, b) = 1$ and*

$$\{n_1, n_2\} \notin \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 2\}\}. \spadesuit$$

§3. Groups generated by $h \geq 3$ powers of twists

In this section the phrase "$i \neq j \neq k$" means that i, j, k are distinct. Let $a_1, \cdots a_h$ be $h \geq 3$ simple closed curves on a surface S such that $(a_i, a_j) > 0$ for $i \neq j$.

Let $\lambda_{ijk} > 1$ and $\mu_{ij} > 0$ (for $i \neq j \neq k$) be real numbers such that $\mu_{ji} = \mu_{ij}^{-1}$. Put $\lambda = (\lambda_{ijk})_{i \neq j \neq k}$ and $\mu = (\mu_{ij})_{i \neq j}$. Define the set of simple closed curves

$$N_{a_i} = N_{a_i, \lambda, \mu} = \{x \mid (x, a_i) < \mu_{ij}(x, a_j), \ \frac{(x, a_k)}{(x, a_j)} < \lambda_{ijk} \frac{(a_i, a_k)}{(a_i, a_j)}, \ \forall j \neq k \neq i\},$$

for $i = 1, \cdots, h$.

Lemma 3.1. *Let $a_1, \cdots, \alpha_h$ be a set of simple closed curves such that $(a_i, a_j) \neq 0$ for $i \neq j$.*

(1) *For any choice of $\lambda_{ijk} \geq 1$, the sets N_{a_i}, $i = 1, \cdots, h$ are mutually disjoint.*

(2) *For $1 \leq i \neq j \leq h$, we have $D_{a_i}^{\pm n}(N_{a_j}) \subset N_{a_i}$ for*

$$\begin{aligned} n \geq \max\{ & \frac{2}{\mu_{ij}(a_i, a_j)}, \\ & \frac{1}{\mu_{ik}(a_i, a_k)} + \lambda_{jik} \frac{(a_j, a_k)}{(a_i, a_j)(a_i, a_k)}, \\ & \frac{\lambda_{jil}}{\lambda_{ikl} - 1} \frac{(a_j, a_l)}{(a_i, a_l)(a_j, a_i)} + \frac{\lambda_{ikl}\lambda_{jik}}{\lambda_{ikl} - 1} \frac{(a_j, a_k)}{(a_j, a_i)(a_i, a_k)}, \\ & \frac{1}{(\lambda_{ikj} - 1)\mu_{ij}(a_i, a_j)} + \frac{\lambda_{ikj}\lambda_{jik}}{\lambda_{ikj} - 1} \frac{(a_j, a_k)}{(a_j, a_i)(a_i, a_k)}, \\ & \frac{\lambda_{ijl}}{(\lambda_{ijl} - 1)\mu_{ij}(a_i, a_j)} + \frac{\lambda_{jil}}{\lambda_{ijl} - 1} \frac{(a_j, a_l)}{(a_j, a_i)(a_i, a_l)} \}_{k \neq l \neq i}. \end{aligned}$$

Proof. (1) is clear. To prove (2), consider $x \in N_{a_j}$. We have

$$(D_{a_i}^{\pm n}(x), a_j) \geq n(a_i, a_j)(x, a_i) - (x, a_j) > \mu_{ji}(x, a_i)$$

for $n \geq \frac{2}{\mu_{ij}(a_i, a_j)}$. Let $k \neq i, j$. Then

$$(D_{a_i}^{\pm n}(x), a_k) \geq n(a_i, a_k)(x, a_i) - (x, a_k) > \mu_{ki}(x, a_i)$$

if

$$n \geq \frac{1}{\mu_{ik}(a_i, a_k)} + \lambda_{jik} \frac{(a_j, a_k)}{(a_i, a_j)(a_i, a_k)}.$$

Let $k, l \neq i$. Then

$$(a_l, D_{a_i}^{\pm n}(x))/(a_k, D_{a_i}^{\pm n}(x)) < \lambda_{ikl}(a_l, a_i)/(a_k, a_i)$$

iff

$$(a_i, a_k)(D_{a_i}^{\pm n}(x), a_l) < \lambda_{ikl}(a_i, a_l)(D_{a_i}^{\pm n}(x), a_k).$$

This will hold if

$$(*) \qquad (a_i, a_k)(n(a_i, a_l)(x, a_i) + (x, a_l)) < \lambda_{ikl}(a_i, a_l)(n(a_i, a_k)(x, a_i) - (x, a_k)).$$

The inequality (*) is equivalent to

$$(**) \qquad n(a_i, a_l)(\lambda_{ikl} - 1) > \frac{(x, a_l)}{(x, a_i)} + \lambda_{ikl}\frac{(x, a_k)(a_i, a_l)}{(x, a_i)(a_i, a_k)}.$$

(one has $(x, a_i) > 0$ since $x \in N_{a_j}$.) Therefore for $l \neq j$ and $k \neq j$ it's enough to have

$$n(a_i, a_l)(\lambda_{ikl} - 1) \geq \lambda_{jil}\frac{(a_j, a_l)}{(a_j, a_i)} + \lambda_{ikl}\lambda_{jik}\frac{(a_j, a_k)(a_i, a_l)}{(a_j, a_i)(a_i, a_k)};$$

i.e.,

$$n \geq \frac{\lambda_{jil}}{\lambda_{ikl} - 1}\frac{(a_j, a_l)}{(a_i, a_l)(a_j, a_i)} + \frac{\lambda_{ikl}\lambda_{jik}}{\lambda_{ikl} - 1}\frac{(a_j, a_k)}{(a_j, a_i)(a_i, a_k)}.$$

If $l = j$ (and so $k \neq j$) then one can replace (**) with

$$n(a_i, a_l)(\lambda_{ikl} - 1) \geq \mu_{ji} + \lambda_{ikl}\frac{(x, a_k)(a_i, a_l)}{(x, a_i)(a_i, a_k)}$$

which gives

$$n \geq \frac{1}{(\lambda_{ikj} - 1)\mu_{ij}(a_i, a_j)} + \frac{\lambda_{ikj}\lambda_{jik}}{\lambda_{ikj} - 1}\frac{(a_j, a_k)}{(a_j, a_i)(a_i, a_k)}.$$

If $k = j$ (and so $l \neq j$) one similarly needs

$$n \geq \frac{\lambda_{ijl}}{(\lambda_{ijl} - 1)\mu_{ij}(a_i, a_j)} + \frac{\lambda_{jil}}{\lambda_{ijl} - 1}\frac{(a_j, a_l)}{(a_j, a_i)(a_i, a_l)}.\spadesuit$$

This lemma conveys the idea that if the set $\{(a_i, a_j)\}_{i \neq j}$ is not "too spread around" then the group $\Gamma = \langle D_{a_1}, \cdots, D_{a_h} \rangle$ is free on n generators, as follows:

Theorem 3.2. *Let $a_1, \cdots, a_h$ be simple closed curves on a surface S such that $M \leq m^2/6$ where $M = \max\{(a_i, a_j)\}_{i \neq j}$ and $m = \min\{(a_i, a_j)\}_{i \neq j}$. Then $\Gamma = \langle D_{a_1}, \cdots, D_{a_h} \rangle \cong \mathbb{F}_h$. More generally, suppose that for all $i \neq j \neq k$ we have*

$$\frac{(a_i, a_k)}{(a_i, a_j)(a_j, a_k)} \leq \frac{1}{6}.$$

Then the same conclusion holds.

Proof. Put $\mu_{ij} = 1$ and $\lambda_{ijk} = 2$ in Lemma 3.1. By assumption, for all $i \neq j \neq k$,

$$\frac{(a_i, a_k)}{(a_i, a_j)(a_j, a_k)} \leq \frac{1}{6}.$$

This implies $(a_i, a_j) \geq 6$ for all $i \neq j$, since otherwise it is impossible for both of

$$\frac{(a_i, a_k)}{(a_i, a_j)(a_j, a_k)} \text{ and } \frac{(a_j, a_k)}{(a_i, a_j)(a_i, a_k)}$$

to be $\leq 1/6$. Therefore, it is easily seen that $n_i = 1$ satisfies the requirements of Lemma 3.1. ♠

On the other hand, if one allows the set $\{(a_i, a_j)\}_{i \neq j}$ to be "more spread around" then one gets weaker results:

Theorem 3.3. *Let $a_1, \cdots, a_h$ be simple closed curves on a surface S such that*

$$M_0 = \max\{\frac{(a_i, a_k)}{(a_i, a_j)(a_j, a_k)}\}_{i,j,k} \geq \frac{1}{6}$$

Then $\Gamma = \langle D_{a_1}^n, \cdots, D_{a_h}^n \rangle \cong \mathbb{F}_h$ for $n \geq 6M_0$.

One can easily construct infinitely many examples of such groups which are non-free, as follows. For a set of simple closed curves $A = \{a_1, \cdots, a_h\}$ define the twist set of A as

$$T(A) = \{g(a) \mid g \in \langle D_\alpha \rangle_{\alpha \in A}, a \in A\}.$$

Now let $A = \{a, b\}$ with $(a, b) = n \geq 2$ and $c_1 \in T(A)$, and pick c such that $(c, c_1) = 1$. (To be able to do this we need c_1 to be non-separating which is possible if at least one of a, b, say a is so.) Then We claim that $\langle D_a, D_b, D_c \rangle \ncong \mathbb{F}_3$. The reason is that since $c_1 = g(a)$ where $g \in \langle D_a, D_b \rangle$, $D_{c_1} = g D_a g^{-1}$. Also D_c, D_{c_1} satisfy the braid relation, which gives

a relation in $\langle D_a, D_b, D_c\rangle$. This, together with Theorem 3.2 above implies that the set of intersection number is "spread around".

Put $A = \{a_1, \cdots, a_h\}$, and

$$X = \{x \ s.c.c. \mid ||x||_A > 0 \ \}.$$

A deficiency of the sets $N_{a_i,\lambda,\mu}$ is that they don't cover X in general. On the other hand, if we set $\lambda_{ijk} = \infty$, for irrational μ_{ij} we get the sets

$$N_{a_i,\mu} = N_{a_i,\infty,\mu} = \{x \mid (x, a_i) < \mu_{ij}(x, a_j)\}$$

which gives a disjoint cover of X, and we can hope to use PPWTC.

Now we try to give some conditions on the a_i and λ in order to have $N_{a_i,\infty,\mu} = N_{a_i,\lambda,\mu}$.

Lemma 3.4. *Suppose for any $i \neq j$, $S_{\{a_i,a_j\}} = S$. Then $N_{a_i,\infty,\mu} = N_{a_i,\lambda,\mu}$ for*

$$\lambda_{ijk} = 2(a_i, a_j)(1 + \frac{(a_k, a_j)}{(a_i, a_k)})(\mu_{ij} + 1).$$

Proof. We only have to show $N_{a_i,\infty,\mu} \subset N_{a_i,\lambda,\mu}$. (The other inclusion is clear.) Suppose $x \in N_{a_i,\infty,\mu}$. Then for $i \neq j \neq k$ we have (using Cauchy-Schwartz)

$$\frac{(x, a_k)}{(x, a_j)} \leq \frac{2((a_i, a_k) + (a_k, a_j))((x, a_i) + (x, a_j))}{(x, a_j)} < 2((a_i, a_k) + (a_k, a_j))(\mu_{ij} + 1).$$

Therefore the lemma follows. ♠

Lemma 3.5. *For $x \in N_{a_j,\lambda,\mu}$ and $i \neq j$ we have $||D^{\pm n}_{a_i}(x)||_A > ||x||_A$ if*

$$n \geq \frac{2}{||a_i||_A}(\mu_{ji} + \sum_{k \neq i,j} \lambda_{jik} \frac{(a_j, a_k)}{(a_j, a_i)})$$

Proof. We have $\sum_k (D^{\pm n}_{a_i}(x), a_k) > \sum_k (x, a_k)$ if

$$n \sum_{k \neq i} (a_i, a_k)(x, a_i) > 2 \sum_{k \neq i} (x, a_k).$$

i.e.,

$$n \sum_k (a_i, a_k) > 2\frac{(x, a_j)}{(x, a_i)} + 2 \sum_{k \neq i,j} \frac{(x, a_k)}{(x, a_i)}.$$

The proof follows. ♠

Putting Lemmas 3.1, 3.4 and 3.5 together we get:

Theorem 3.6. *Let $\{a_1, \cdots, a_h\}$ be a set of simple closed curves such that $S_{\{a_i,a_j\}} = S$ for $i \neq j$. Then $G = \langle D^n_{a_1}, \cdots, D^n_{a_h}\rangle$ is relatively pseudo-Anosov if $n \geq \max\{6M/m, 4M/m+5\}$, where $m = \min I \geq 2$ and $M = \max I$ in which $I = \{(a_i, a_j) | i \neq j\}$.*

Proof. Put $\mu_{ij} = 1$ and take λ_{ijk} as given by Lemma 3.4. A fairly straightforward computation shows that the number n given satisfies the requirement of 3.5 and the first two of 3.1 ♠

It would be nice to improve this result. See Question 0.2.

§4 Example: A group generated by 3 twists

To show that how the sets $N_{a_j,\lambda,\mu}$ work, let's look at the case a_1, a_2, a_3 are simple closed curves such that $(a_i, a_j) = 1$ for $i \neq j$ on a torus. The group $G = G(n_1, n_2, n_3) = \langle D^{n_1}_{a_1}, D^{n_2}_{a_2}, D^{n_3}_{a_3}\rangle$ was studied by Backmuth, Mochizuki, and Scharlemann ([BM], [Sch]) in the case of real exponents n_i. The result of [Sch] implies that G is free on 3 generators if

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \leq 1.$$

Here we show how the results of §3 imply this. Without loss of generality we may assume $a_1 = (1, 0)$, $a_2 = (0, 1)$, and $a_3 = (1, 1)$. Then, we are studying the groups

$$G = \langle \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{n_1}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{n_2}, \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^{n_3} \rangle$$

Since a_i cut the torus into two triangles, one can easily see that for any simple closed curve x,

$$(x, a_i) = (x, a_j) + (x, a_k)$$

for some $\{i, j, k\} = \{1, 2, 3\}$. Suppose the numbers μ_{ij} satisfy the inequalities $\mu_{ji} + \mu_{ki} \geq 1$. Now if for example $(x, a_1) < \mu_{12}(x, a_2)$ and $(x, a_1) < \mu_{13}(x, a_3)$ then either $(x, a_2) = (x, a_1) + (x, a_3)$ which implies

$$1 \leq \frac{(x, a_2)}{(x, a_3)} < 1 + \mu_{13}$$

or, $(x, a_3) = (x, a_1) + (x, a_2)$ which implies

$$1 \leq \frac{(x, a_3)}{(x, a_2)} < 1 + \mu_{12}.$$

(One cannot have $(x,a_1) = (x,a_2) + (x,a_3)$ since this implies $\mu_{21} + \mu_{31} < 1$.) This means that, $N_{a_i,\lambda,\mu} = N_{a_i,\infty,\mu}$ for

$$\lambda_{ijk} = 1 + \mu_{ij}.$$

So one need not worry about making sure the conditions

$$(x,a_k)/(x,a_j) < \lambda_{ijk}(a_i,a_k)/(a_i,a_j)$$

are satisfied. This together with Lemma 3.1 shows that $G(n_1,n_2,n_3) \cong \mathbb{F}_3$ if

$$n_i \geq \max\{2\mu_{ji}, 1 + \mu_{ji} + \mu_{ki}\}_{j \neq k \neq i}.$$

Now putting $\mu_{ij} = 1$ implies that $G \cong \mathbb{F}_3$ if $n_i \geq 3$ for all i.

On the other hand, if we assume $\mu_{21} = \mu_{31} = 1/2$ and $\mu_{32} = 1$ we get that G is free if $n_1 = 2$ and $n_2, n_3 \geq 4$.

Lastly, assume $\mu_{21} = 2/3$, $\mu_{31} = 1/3$ and $\mu_{32} = 1/2$. Then one gets that G is free for $n_1 = 2$, $n_2 = 3$ and $n_3 \geq 6$.

This shows that $G \cong \mathbb{F}_3$ if $1/n_1 + 1/n_2 + 1/n_3 \leq 1$.

To see for what values of n_i the group G is relatively Anosov we could perturb the numbers $\mu_{21}, \mu_{31}, \mu_{32}$ to be irrational:

Theorem 4.1. *Let G be the group $\langle D_{a_1}^{n_1}, D_{a_2}^{n_2}, D_{a_3}^{n_3} \rangle$ where a_i are simple closed curves on a torus or punctured torus with $(a_i,a_j) = 1$ for $i \neq j$ and n_i are positive integers.*

(1) *$G \cong \mathbb{F}_3$ if $1/n_1 + 1/n_2 + 1/n_3 \leq 1$.*
(2) *G is relatively Anosov if $1/n_1 + 1/n_2 + 1/n_3 < 1$.*

Proof. (1) was already proved. To prove (2), notice that the situation is completely symmetric so one can assume that $a = a_1 = (1,0)$, $b = a_2 = (0,1)$ and $c = a_3 = (1,1)$. PPWTC shows that for any choice of μ_{ij} such that all three of $\mu_{21}, \mu_{31}, \mu_{32}$ are irrational and $\mu_{ji} + \mu_{ki} \geq 1$ for $i \neq j \neq k$, G is Anosov if

$$n_i \geq \max_j\{1 + 2\mu_{ji}\} = \max_{j \neq k}\{2\mu_{ji}, 1 + \mu_{ji} + \mu_{ki}, 1 + 2\mu_{ji}\}.$$

Therefore, if $\mu_{21} = \mu_{31} = 1 - \epsilon$ and $\mu_{32} = 1 + \epsilon$ where ϵ is a small irrational, then we get the condition $n_1 \geq 3$ and $n_2, n_3 \geq 4$.

Similarly if $\mu_{21} = 2/3+\epsilon$ and $\mu_{31} = 1/3-\epsilon$ and $\mu_{32} = 1/2+\epsilon$, then we get the condition $n_1 \geq 2$ and $n_2 \geq 4$ and $n_3 \geq 8$.

This means that G is not relatively pseudo-Anosov only possibly When

$$\{n_1, n_2, n_3\} = \{1, *, *\}, \{2, 2, *\}, \{2, 3, *\}, \{3, 3, *\}, \{2, 4, n\},\ 4 \leq n \leq 7.$$

One can cut this list a little short by using WPP, as follows: With $\mu_{ij} = 1$ the only curves not covered by N_i are $(1, -1)$ and $(2, 1)$. Then one can actually check that for $(n_1, n_2, n_3) = (3, 3, n)$, $n \geq 4$, WPP can be applied with $n_0 = 3$.

Also, in the case $(n_1, n_2, n_3) = (2, 3, n)$, $n > 7$, considering $\mu_{21} = 2/3$, $\mu_{31} = 1/3$, and $\mu_{32} = 1/2$, one can see that only the curves $(2, 3)$, $(2, -3)$, $(4, 3)$ are not covered by N_i. Now WPP can be applied with $n_0 = 3$.

Similarly for $(n_1, n_2, n_3) = (2, 4, n)$, $n > 4$, one can use WPP.

So the list of possible exceptions cuts down to

$$\{1, *, , *\}, \{2, 2, *\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 4, 4\}, \{3, 3, 3\}.$$

This finishes the proof of the Theorem. ♠

4.2. Remark. The number 1 is the best possible in the above theorem, since $D_b^4 D_a^2 D_c^4$ fixes $(1, 2)$ and $D_b^3 D_a^2 D_c^6$ fixes $(2, 3)$. Also, $D_b^2 D_a^2 D_c^n$ fixes $(1, 1)$, so this element commutes with D_c.

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